Proof of the duality thm

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(following Weselmann)

0. Recap:

G reductive grp $/ k = \bar{k}$ $\rightsquigarrow$ $\mathcal{P}$ Picard stack of $\mathcal{J}$-torsors

$p = \operatorname{char} k \nmid |W|$

$$\begin{array}{c} \mathcal{P} \\ \downarrow \\ \mathcal{B}^\circ \end{array}$$

$\check{G}$ Langlands dual $\rightsquigarrow$ $\check{\mathcal{P}}$ over $\check{\mathcal{B}}^\circ \cong \mathcal{B}^\circ$

<u>Goal:</u> $\exists$ natural iso of Picard stacks $\mathcal{P}^\vee \xrightarrow{\sim} \check{\mathcal{P}}$,
where $\mathcal{P}^\vee := \mathcal{H}om_{PS/\mathcal{B}^\circ}(\mathcal{P}, B\mathbb{G}_m)$.

Have already reduced this to the corresponding claim for

$P :=$ coarse moduli space of $\mathcal{P}_b^\circ$, $b \in \mathcal{B}^\circ(k)$.

Writing $J := J_b$, have isogenies

$$\begin{array}{ccccc} P^\circ & \longrightarrow & P & \longrightarrow & P^1 \\ \| & & \| & & \| \\ H^1(C, J^\circ)^\circ & & H^1(C, J)^\circ & & H^1(C, J^1)^\circ \end{array}$$

$J^1 := (\pi_* T)^W$
$\cup$
J
$\cup$
$J^\circ :=$ fibrewise conn. cpt of J^1

Counting argument has reduced us to the following

<u>Thm 1</u>: $\exists$ natural iso

$$\begin{tikzcd} (\check{P}^1)^\vee \arrow[r, "\sim"] \arrow[d] & P^0 \arrow[d] \\ (\check{P})^\vee \arrow[r, "\mathcal{D}"] & P \end{tikzcd}$$

lifting the duality map

($\Rightarrow \mathcal{D}$ is also an iso)

Explicitly,

$$\check{P}^1 = H^1(C, \check{J}^1) = (\mathrm{Jac} \otimes X^*)^{W,0}$$

$\mathrm{Jac} = \mathrm{Jac}\, \tilde{C}$

$X^* = X^*(T)$
$= X_*(\check{T})$

$$\Rightarrow (\check{P}^1)^\vee = (\mathrm{Jac} \otimes X_*)_W$$

and the norm map induces a natural map

$$\begin{tikzcd} \mathrm{Jac} \otimes X_* = H^1(C, \pi_* T)^0 \arrow[r, "\mathrm{Nm}"] \arrow[dr, two heads] & H^1(C; J^0)^0 = P^0 \\ & (\mathrm{Jac} \otimes X_*)_W = (\check{P}^1)^\vee \arrow[u, dashed] \end{tikzcd}$$

& we want this to be an iso.

Prime-to-p-isogeny $\Rightarrow$ enough to show

<u>**Prop 2**</u>: $(\mathrm{Jac} \otimes X_*)_W[n] \twoheadrightarrow P^0[n]$ epi $\forall n$ with $p \nmid n$.

1. Connected components & Kummer sequences

Have

$$0 \to P^0[n] \to H^1(C, J^0)[n] \to \pi_0(P^0)[n] \to 0$$

$$\pi_0(P^0)[n] \overset{\text{Ngo, Faltings}}{=} (X_*)_W[n]$$

$\Rightarrow$ enough to show

$$|\operatorname{coke}((Jac \otimes X_*)_W[n] \to H^1(C, J^0)[n])| \leq |(X_*)_W[n]|.$$

Reformulate this:

Kummer sequence for $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1$
and $H^1(\tilde{C}, T)[n] = (Jac \otimes X_*)[n]$ gives

$$0 \to H^1(\tilde{C}, \mu_n \otimes X_*) \to Jac \otimes X_* \xrightarrow{n} Jac \otimes X_*$$

and since $(-)_W$ is right exact, we get:

$$H^1(\tilde{C}, \mu_n \otimes X_*)_W \twoheadrightarrow (Jac \otimes X_*)_W[n].$$

Kummer sequence for $0 \to J^0[n] \to J^0 \to J^0 \to 0$

gives
$$H^1(C, J^0[n]) \twoheadrightarrow H^1(C, J^0)[n].$$

$\Longrightarrow$ enough to show

Prop. 3
$$\Big|\operatorname{cok}\Big(\underbrace{H^1(\tilde{C}, \mu_n \otimes X_*)_W}_{= H^1(C, \pi_*(\mu_n \otimes X_*))} \xrightarrow{\alpha} H^1(C, J^0[n])\Big)\Big| \leq |(X_*)_W[n]|.$$

Here α is induced by the norm map

$$\begin{tikzcd} \eta : \pi_*(\mu_n \otimes X_*) \arrow[r, two heads] \arrow[rr, bend right] & \pi_*(\mu_n \otimes X_*)_W \arrow[r] & J^0[n] \end{tikzcd}$$

Note: ν is an epi on $U = C \setminus \bigcup_\alpha C^\alpha$
but has nontrivial cokernel on C^α!

2. An application of Poincaré duality

Consider

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ 0 \arrow[r] & \ker(\eta) \arrow[r] & \pi_*(\mu_n \otimes X_*) \arrow[r] & \operatorname{im}(\eta) \arrow[r] \arrow[d] & 0 \\ & & & J^0[n] \arrow[d] & \\ & & & \operatorname{cok}(\eta) \arrow[d] & \\ & & & 0 & \end{tikzcd}$$

Get

$$\begin{tikzcd}
 & H^0(C, \mathrm{cok}(\eta)) \arrow[d] \arrow[dr, "\beta"] & & \\
H^1(C, \pi_*(\mu_n \otimes X_*)) \arrow[r] \arrow[dr, "\alpha"'] & H^1(C, \mathrm{Im}(\eta)) \arrow[r] \arrow[d] & K \arrow[r] & 0 \\
 & H^1(C, \mathcal{J}^0[n]) & &
\end{tikzcd}$$

for $\mathbf{K} := \ker\left(H^2(C, \ker(\eta)) \xrightarrow{\gamma} H^2(C, \pi_*(\mu_n \otimes X_*))\right)$

$\Longrightarrow \mathrm{cok}(\alpha) \simeq \mathrm{cok}(\beta)$

<u>Lemma 4.</u> $K^\vee \cong H^1(W, \mathbb{Z}/n\mathbb{Z} \otimes X^\circ)$.

Proof. Let $j: U \hookrightarrow C$.

$H^2(C, \ker(\eta)) = H^2_c(U, \ker \eta|_U)$ since $H^i(C, (\ldots)/_{j_! j^*(\ldots)}) = 0$ $\forall i > 0$

$H^2(C, \pi_*(\mu_n \otimes X_*)) = H^2_c(U, \ldots)$

Poincaré duality:

$$K^\vee \cong \mathrm{cok}\Big(\underbrace{H^0(U, \pi_*(\mathbb{Z}/n\mathbb{Z} \otimes X^*))}_{\mathbb{Z}/n\mathbb{Z} \otimes X^*} \xrightarrow{\gamma^\vee} \underbrace{H^0(U, \ker(\eta)^\vee|_U)}_{(\ker(\eta)^\vee_u)^{\pi_1(U,u)}} \Big)$$

Recall

acting diagonally

$$
\begin{tikzcd}
0 \arrow[r] & \left(\mathbb{Z}/n\mathbb{Z}[W] \otimes X^*\right)^W \arrow[r] \arrow[d, phantom, "\parallel"] & \mathbb{Z}/n\mathbb{Z}[W] \otimes X^* \arrow[r] \arrow[d, phantom, "\wr\wr"] & \ker(\eta)^\vee_u \arrow[r] & 0 \\
& \underbrace{\left(\pi_*(\mu_n \otimes X_*)_W\right)^\vee_u}_{(\simeq\, J^0[n] \text{ over } U!)} & \pi_*(\mu_n \otimes X_*)^\vee_u & &
\end{tikzcd}
$$

The $\pi_1(U,u)$-action on $\mathbb{Z}/n\mathbb{Z}[W] \otimes X^*$ is the W-action on the first factor (not diagonal).

Under $\left(\mathbb{Z}/n\mathbb{Z}[W] \otimes X^*\right)^W \simeq \mathbb{Z}/n\mathbb{Z} \otimes X^*$ this induces the natural W-action on X^*.

$\Longrightarrow$ get by applying invariants under $\pi_1 = \pi_1(U,u)$:

$$
\begin{tikzcd}
0 \arrow[r] & \left(\mathbb{Z}/n\mathbb{Z} \otimes X^*\right)^{\pi_1} \arrow[r] & \left(\mathbb{Z}/n\mathbb{Z}[W] \otimes X^*\right)^{\pi_1} \arrow[r, "\gamma^\vee"] \arrow[d, phantom, "\parallel"] & \left(\ker(\eta)^\vee_u\right)^{\pi_1} \\
& & \mathbb{Z}/n\mathbb{Z} \otimes X^* & H^1(W, \mathbb{Z}/n\mathbb{Z} \otimes X^*) \\
& & & H^1(W, \mathbb{Z}/n\mathbb{Z}[W] \otimes X^*) = 0
\end{tikzcd}
$$

(W-action only on $\mathbb{Z}/n\mathbb{Z}[W]$, not on X^*)

$$\Longrightarrow \quad K^\vee = \operatorname{cok}(\gamma^\vee) \overset{!}{=} H^1(W, \mathbb{Z}/n\mathbb{Z} \otimes X^*). \qquad \square$$

3. The dual of β

Want to control kernel of

$$\beta^\vee : K^\vee = H^1(W, \mathbb{Z}/n\mathbb{Z} \otimes X^*) \longrightarrow H^0(C, \operatorname{cok}(\eta))^\vee.$$

<u>Lemma 5.</u> $H^0(C, \operatorname{cok}(\eta)) = \prod_{\alpha \in W \setminus \Phi} \prod_{x \in C^\alpha} \operatorname{cok}(\eta)_x$

where $$\operatorname{cok}(\eta)_x = \frac{\ker(\alpha_{red} : \mu_n \otimes X_\circ \to \mu_n)}{(1 + s_\alpha)(\mu_n \otimes X_\circ)}$$

(nontrivial only if $2 \mid n$)

Proof. $J^\circ[n]_x \cong \ker(\alpha_{red} \mid_{\mu_n \otimes X^\circ})$ for $x \in C^\alpha$
& image of Norm map on stalks above x is given by product of "local norms" $1 + s_\alpha$. $\square$

<u>Lemma 6.</u> The natural pairing $(\mathbb{Z}/n\mathbb{Z} \otimes X^*) \times (\mu_n \otimes X_*) \longrightarrow \mu_n$

descends to

$$\beta^\vee : H^1(W, \mathbb{Z}/n\mathbb{Z} \otimes X^*) \times H^0(C, \operatorname{cok}(\eta)) \longrightarrow \mu_n.$$

Proof.

$$(\varphi : W \to \mathbb{Z}/n\mathbb{Z} \otimes X^\circ) \in H^1(W, \mathbb{Z}/n\mathbb{Z} \otimes X^\circ)$$

$$y = (y_\alpha \in \ker(\alpha_{red}))_{\substack{\alpha \in W\backslash\Phi \\ x \in C^\alpha}} \in H^0(C, \operatorname{cok}(\eta^\vee))$$

$$\rightsquigarrow$$

$$\langle \varphi, y \rangle := \prod_{\substack{\alpha \in W\backslash\Phi \\ x \in C^\alpha}} \langle \varphi(s_\alpha), y_\alpha \rangle$$

Cocycle relations $\Rightarrow$ this is well-defined, doesn't depend on chosen representatives for $W\backslash\Phi$

To see this is $= \beta^\vee$, use a Čech calculation $/\mathbb{C}$ & specialization to char p $\square$

Prop 3 (and hence thm 1) now follow from:

<u>Lemma 7.</u> $\ker(\beta^\vee) \simeq \operatorname{cok}(\mathbb{Z}/n\mathbb{Z} \otimes X^\circ \xrightarrow{\delta} (\mathbb{Z}/n\mathbb{Z})^\Delta)$

$$z \longmapsto (\langle \check{\alpha}_{ext}, z \rangle)_{\alpha \in \Delta}$$

where $\check{\alpha}_{ext} := m_\alpha \cdot \check{\alpha}$

with $\alpha(X_\circ) =: m_\alpha \cdot \mathbb{Z}$

& this is dual to

$$(X_\circ)_W[n] \simeq \operatorname{cok}(\mathbb{Z}^\Delta \xrightarrow{\varepsilon} X_\circ)[n]$$

$$(n_\alpha)_{\alpha \in \Delta} \mapsto \sum_\alpha n_\alpha \cdot \check{\alpha}_{ext}.$$

Proof.

(a) By lemma 6, a cocycle $\varphi : W \to \mathbb{Z}/n\mathbb{Z} \otimes X^{\circ}$ represents a class in $\ker(\beta^{\vee})$ iff $\varphi(s_\alpha) = n_\alpha \cdot \alpha_{red}$ $\forall \alpha \in \Phi$.

Such cocycles are in bijection with $(\mathbb{Z}/n\mathbb{Z})^{\Delta}$

via $\varphi \mapsto (n_\alpha)_{\alpha \in \Delta}$

("$\hookrightarrow$" clear, "$\twoheadrightarrow$" follows by verifying $\varphi(r) = 0$ for Coxeter relations r)

If $\varphi = \partial z$ is a coboundary of $z \in \mathbb{Z}/n\mathbb{Z} \otimes X^{\circ}$

then
$$\varphi(s_\alpha) = (1 - s_\alpha)(z) = \alpha \cdot \langle \check{\alpha}, z \rangle$$
$$= \underbrace{\langle \check{\alpha}_{ext}, z \rangle}_{n_\alpha} \cdot \alpha_{red}$$

& conversely. So $\ker(\beta^{\vee}) \simeq \operatorname{cok}(\delta)$

(b) Clearly $(X_\circ)_W \simeq X_\circ \Big/ \sum_{\alpha \in \Delta} \underbrace{(1 - s_\alpha) . X_\circ}_{= \check{\alpha}_{ext} \cdot \mathbb{Z}}$ (the s_α, $\alpha \in \Delta$, generate W)

$\simeq \operatorname{cok}(\varepsilon)$.

Now

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^{\Delta} \arrow[r, "\varepsilon"] \arrow[d, "n"] & X_{\circ} \arrow[r] \arrow[d, "n"] & (X_{\circ})_{w} \arrow[r] \arrow[d, "n"] & 0 \\ 0 \arrow[r] & \mathbb{Z}^{\Delta} \arrow[r, "\varepsilon"] & X_{\circ} \arrow[r] & (X_{\circ})_{w} \arrow[r] & 0 \end{tikzcd}$$

implies:

$$\mathrm{cok}(\varepsilon)[n] \simeq \ker\left(\mathbb{Z}^{\Delta} \otimes \mathbb{Z}/n\mathbb{Z} \xrightarrow{\varepsilon} X_{\circ} \otimes \mathbb{Z}/n\mathbb{Z}\right),$$

which is dual to $\mathrm{cok}(\delta)$. $\square$